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CONIC SECTIONS.

THE METHOD OF PROJECTIONS.

BY

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EDWARD STANFORD, 55, CHARING CROSS, S.W.

1877.

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PREFACE.

THIS treatise is designed to furnish a simple explanation of the method of projections, which at once powerful and elegant has no special difficulty that it need be deferred to the higher parts of Conic Sections.

The demonstrations, by means of this principle, of some of the chief properties of the ellipse will, it is hoped, sufficiently illustrate its use: a few subsidiary points are noticed in an appendix.

The theory of infinitesimals has been in general avoided, but an attempt is made in Proposition XXIX. to remove the difficulty it so often presents, and Example 16 is a rider to it.

Questions involving foci, directrices, or angles can only be treated indirectly, and rarely with any advantage, by projections, and Propositions XXIII. and XXV. are not required for any of the problems here given.

The examples have been carefully selected from college, school, and other papers.

S. B. K.

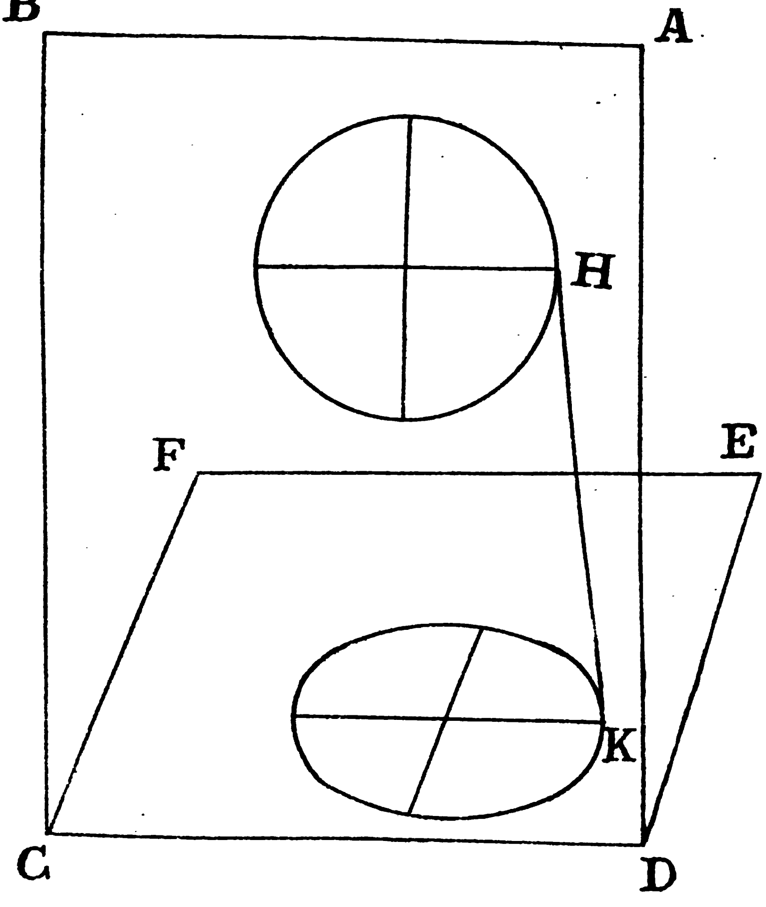
IPSWICH SCHOOL, *February*, 1877.

DEFINITION.

Let H be a line in the plane A B C D, and let a straight line H K move along H so as to be always perpendicular to the plane C D E F: the line traced out by its foot K on the plane C D E F is the orthogonal projection of H on that plane.

In this book the corresponding points are indicated by the same letters in both planes, but the letters which refer to the figure in the plane A B C D, i. e. that of which the one under consideration is a projection, are accented; thus H in the above definition would be distinguished in the following propositions as K′.

Since conical projection is not here treated of, the word projection is always to be understood to mean orthogonal projection as defined above.


B
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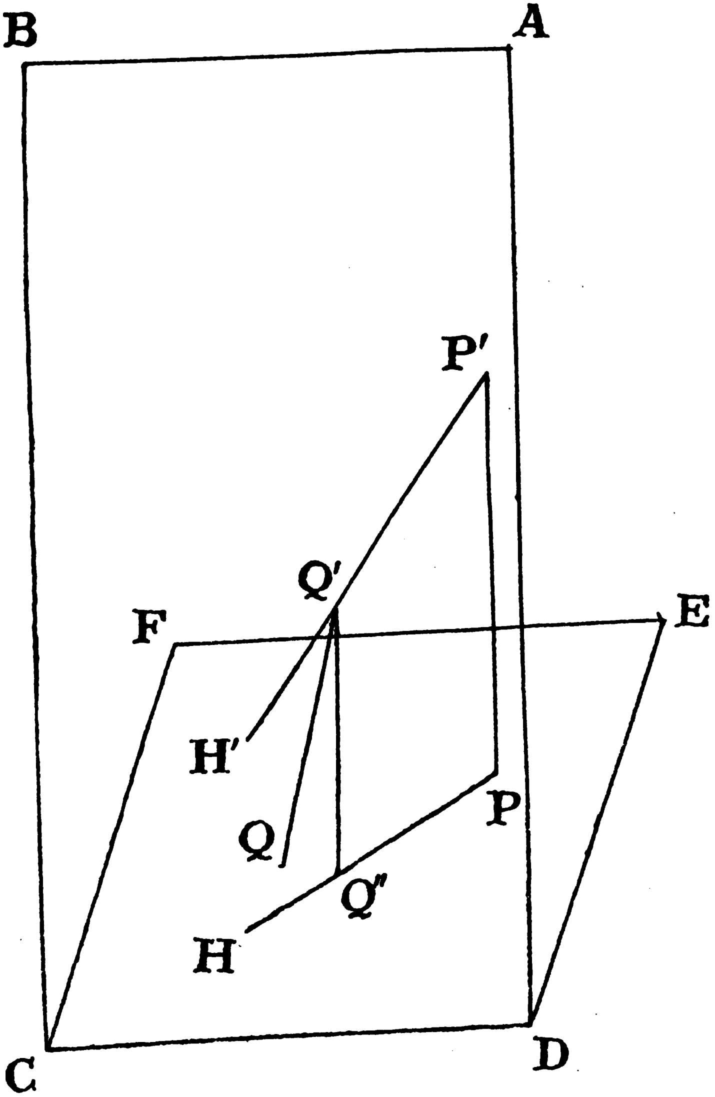
C
D


B
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P′
Q′
F
E
H′
Q
P
Q″
H
C
D

PROPOSITION I.

A straight line projects into a straight line.

Let H′ be a straight line in the plane A B C D, P′ any point in H′, P the projection of P′ on the plane C D E F.

Let a plane pass through H′ and P′ P (Eu. XI. 2) cutting the plane C D E F in H; from Eu. XI. 3, H is a straight line.

If possible let the projection of Q′ another point in H′ meet the plane C D E F in a point Q without H.

From Q′ draw Q′ Q″ perpendicular to H, in the plane of H′ and P′ P (Eu. I. 12);

from Eu. XI. 8, Q′ Q″ is perpendicular to the plane C D E F;

∴ from the point Q′ are drawn two perpendiculars to the plane C D E F, which is impossible (Eu. XI. 13);

∴ Q is in the straight line H.

PROPOSITION II.

Parallel straight lines project into parallel straight lines.

Let $H'H_1'$, $K'K_1'$ be two parallel straight lines;

HH_1, KK_1 their projections;

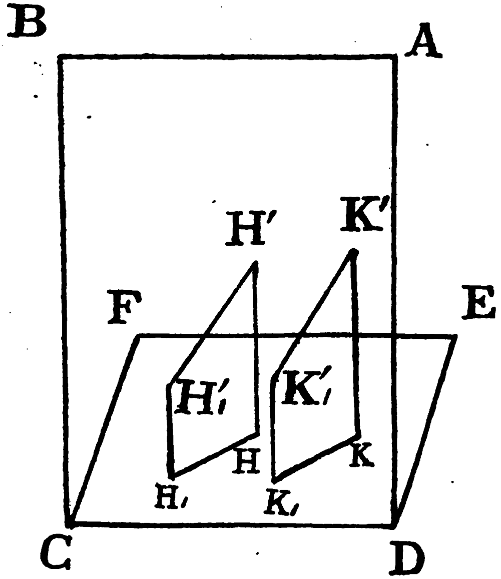
$H'H$, $K'K$ their projectors at any moment.

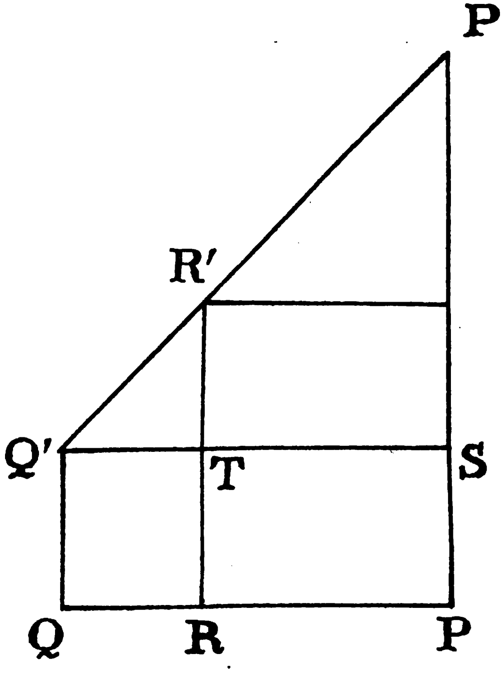
From Eu. xi. 6, $H'H$ $K'K$ are parallel;

∴ the planes through $H'H$ $H'H_1'$, and $K'K$ $K'K_1'$ are parallel (Eu. xi. 15).

From Proposition I., the intersections of these planes with the plane CDEF are the projections of $H'H_1'$, $K'K_1'$;

from Eu. xi. 16 these sections are parallel.


B
A
H′
K′
F
E
H′,
K′,
H
K
H,
K,
C
D


P′
R′
Q′
T
S
Q
R
P

PROPOSITION III.

If a straight line be divided at any point into two parts, the projection of this point will divide the projection of the straight line into two parts having the same ratio to one another which the former parts have.

Let P′ Q′ be divided in R′.

Let P Q be the projection of P′ Q′, R of R′.

In the plane of P′ Q′, P Q, P′ P, R′ R, draw through Q′ Q′ S parallel to Q P, and cutting R′ R in T, and P′ P in S.

∴ (Eu. vi. 2) Q′ T : T S : : Q′ R′ : R′ P′;

from Eu. i. 34, Q′ T = Q R, T S = R P;

∴ Q R : R P : : Q′ R′ : R′ P′.

PROPOSITION IV.

The projections of parallel straight lines have to one another the same ratio which the lines themselves have to one another.

Let A′B′, C′D′ be parallel straight lines; AB, CD their projections.

A′B′, B′B are in one plane (Eu. XI. 6); as are also similarly C′D′, D′D.

Through B′ draw MB′, parallel to AB; and through C′ draw NC′, parallel to CD.

The angle at B′ = the angle D′ (Eu. XI. 10);

∴, since the angles at M and N are right angles, the triangle A′B′M is similar to C′D′N;

$$\therefore\ B'A' : B'M :: C'D' : D'N;$$

from Eu. I. 34, B′M = BA, D′N = CD;

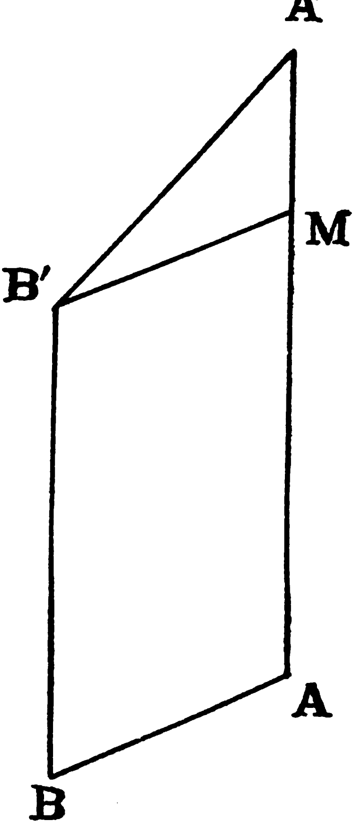
$$\therefore\ A'B' : AB :: C'D' : CD;$$

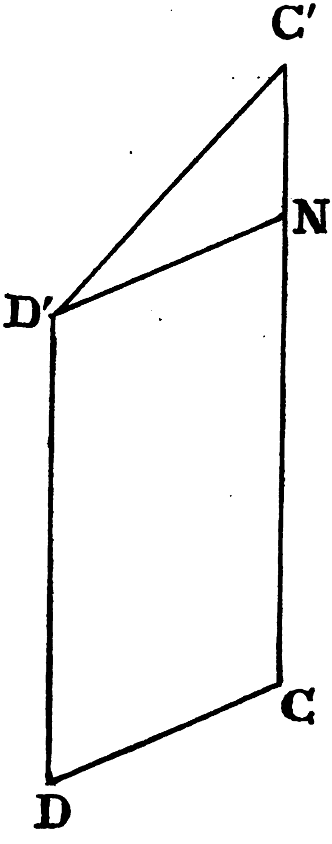
alternando, $A'B' : C'D' :: AB : CD.$

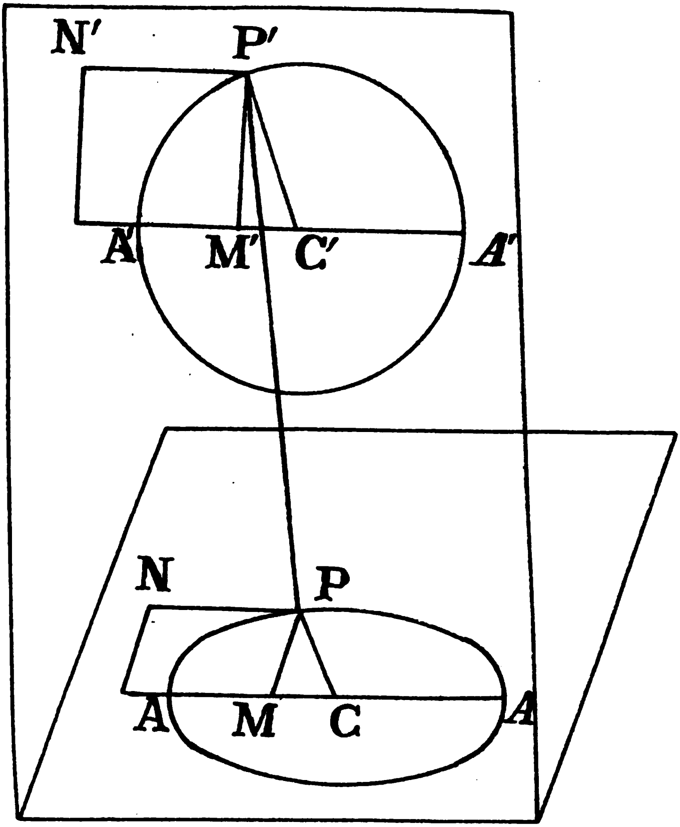
DEFINITION.

An ellipse is a plane curve such that the distance of any point of it from a given fixed point bears a constant ratio, less than unity, to its distance from a given fixed straight line.

The fixed point is called its Focus; the fixed line its Directrix.


A′
M
B′
A
B


C′
N
D′
C
D


N′
P′
A′
M′
C′
A′
N
P
A
M
C
A

PROPOSITION V.

A circle projects into an ellipse.

Let P′ be a point on the circumference of the circle; P its projection; A *A* the projection of the diameter of the circle parallel to the line of intersection of the plane of the circle with the plane of the ellipse; P′ M′, P M ordinates of P′ and P respectively at right angles to the diam. and its projection.

Let the ratio P M : P′ M′ be denoted by $b : a$;

$$\therefore PM^2 : P'M'^2 :: b^2 : a^2;$$

$$P'M'^2 = C'P'^2 - C'M'^2 \text{ (Eu. I. 47) } = C'P'^2 - CM^2;$$

$$\therefore PM^2 : C'P'^2 - CM^2 :: b^2 : a^2;$$

$$\therefore PM^2 : CA^2 - CM^2 :: b^2 : a^2.$$

Take C″ A″ so that

$$C''A''^2 : CA^2 :: b^2 : a^2;$$

and C″ M″ so that

$$C''M''^2 : CM^2 :: b^2 : a^2;$$

$$\therefore C''A''^2 - C''M''^2 : CA^2 - CM^2 :: b^2 : a^2;$$

$$\therefore PM^2 = C''A''^2 - C''M''^2.$$

Take S in the straight line A *A*, so that

$$C'M'^2 : CM^2 :: CS^2 : CA^2 :: a^2 - b^2 : a^2;$$

$$\therefore CM^2 - C'M'^2 : CM^2 :: b^2 : a^2 :: CA^2 - CS^2 : CA^2;$$

$$\therefore C''A''^2 = CA^2 - CS^2,\ C''M''^2 = CM^2 - C'M'^2;$$

$$\begin{aligned}\therefore PM^2 &= CA^2 - CS^2 - (C''M''^2 - C'M'^2)\\ &= CA^2 + C'M'^2 - (CM^2 + CS^2). \quad \text{(i)}\end{aligned}$$

Since $\mathrm{CS} : \mathrm{CA} :: \mathrm{C'M'} : \mathrm{CM}$,

$$2\,\mathrm{CS} \, . \, \mathrm{CM} = 2\,\mathrm{CA} \, . \, \mathrm{C'M'};$$

adding this to and subtracting it from each side of (i),

$$\begin{aligned} \mathrm{PM}^2 &= (\mathrm{CA}^2 - 2\,\mathrm{CA} \, . \, \mathrm{C'M'} + \mathrm{C'M'}^2) \\ &\qquad - (\mathrm{CM}^2 - 2\,\mathrm{CM} \, . \, \mathrm{CS} + \mathrm{CS}^2), \\ &= (\mathrm{CA} - \mathrm{C'M'})^2 - (\mathrm{CM} - \mathrm{CS})^2 \quad \text{(corol. to Eu. II. 7)}, \\ &= (\mathrm{CA} - \mathrm{C'M'})^2 - \mathrm{SM}^2; \end{aligned}$$

$$\therefore\ \mathrm{SP}^2 = \mathrm{PM}^2 + \mathrm{SM}^2 = (\mathrm{CA} - \mathrm{C'M'})^2;$$

$$\therefore\ \mathrm{SP} = \mathrm{CA} - \mathrm{C'M'}.$$

Take O in M A produced, so that

$$\mathrm{CS} : \mathrm{CA} :: \mathrm{CA} : \mathrm{CO};$$

$$\therefore\ \mathrm{CA}^2 : \mathrm{CO}^2 :: \mathrm{C'M'}^2 : \mathrm{CM}^2 :: a^2 - b^2 : a^2;$$

$$\therefore\ \mathrm{CA} : \mathrm{CO} :: \mathrm{C'M'} : \mathrm{CM};$$

$$\therefore\ \mathrm{CA} : \mathrm{C'M'} :: \mathrm{CO} : \mathrm{CM};$$

$$\therefore\ \mathrm{CA} - \mathrm{C'M'} : \mathrm{CO} - \mathrm{CM} :: \mathrm{CA} : \mathrm{CO}.$$

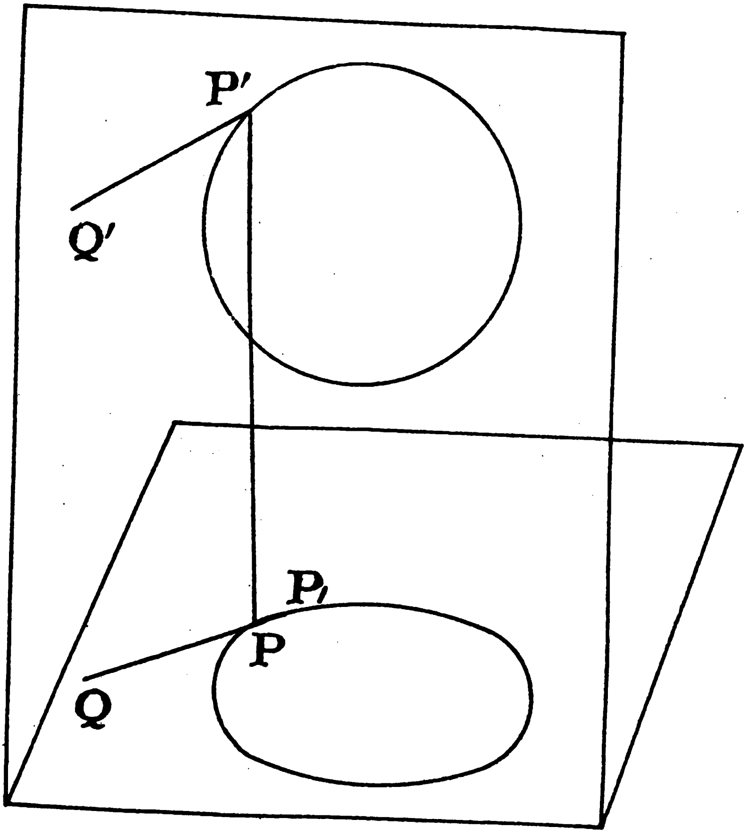
Noting that $\mathrm{CO} - \mathrm{CM} = \mathrm{OM} = \mathrm{PN}$, where P N is the perpendicular from P onto the straight line O N drawn through O at right angles to O C,

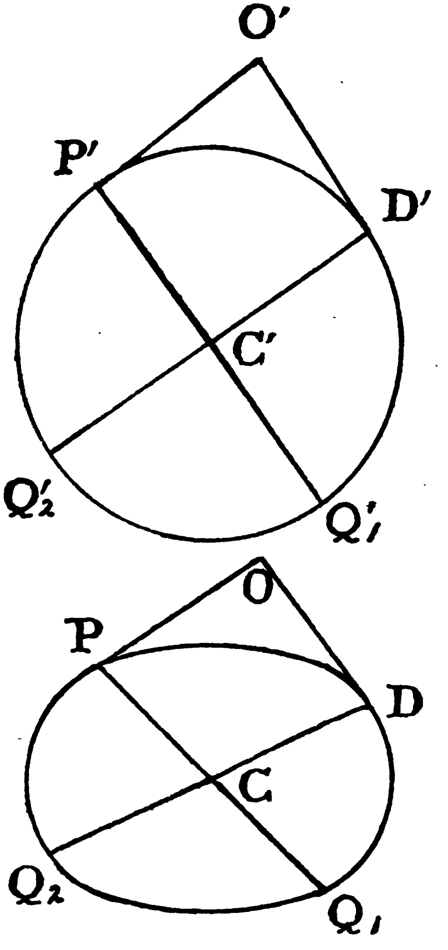
$$\mathrm{PS} : \mathrm{PN} :: \mathrm{CA} : \mathrm{CO}.$$

That is, the distance of P from the focus S bears a constant ratio to its distance from the directrix N O, and this ratio, since S A is less than S O, is less than unity.

PROPOSITION VI.

The tangent at any point of the circle of projection projects into the tangent at the corresponding point of the ellipse.


P′
Q′
P₁
P
Q


O′
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D′
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Q′₂
Q′₁
O
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C
Q₂
Q₁

Let P′ Q′ be the straight line touching the circle of projection in P′.

P, the projection of P′, is a point on the ellipse and also on P Q the projection of P′ Q′;

$\therefore$ P Q meets the ellipse in P.

It does not cut it.

If possible, let it cut it in P_1;

P_1 is a point on the ellipse,

$\therefore$ it is the projection of a point P_1' on the circle of projection;

and since Q′ is also a point on the straight line P′ Q′ of which P Q is the projection;

$\therefore$ P′ Q′ cuts the circle.

But, by supposition, P′ Q′ does not cut the circle, but meets it only in one point P′.

$\therefore$ P Q does not cut the ellipse, but meets it only in the one point P.

PROPOSITION VII.

Two tangents can be drawn to an ellipse from any point without it.

This point O is evidently the projection of some point O′ without the circle of projection.

From O′ two tangents can be drawn to the circle; the projection of each of these is, from Proposition VI., a tangent to the ellipse.

DEFINITIONS.

The AXIS MAJOR is the line A *A*, which passes through the CENTRE C and the FOCUS S of the ellipse.

The AXIS MINOR is the shorter axis, drawn through C at right angles to A *A*.

The LATUS RECTUM is the straight line drawn through the focus S at right angles to the axis major and terminated both ways by the ellipse.

A DIAMETER of an ellipse is a straight line passing through the centre and terminated both ways by the circumference.

If $P_1 P_2$, $D_1 D_2$ be two diameters such that $P_1 P_2$ is parallel to the tangent to the ellipse at the point D_1, the extremity of the other diameter, the diameter $P_1 P_2$ is said to be CONJUGATE to $D_1 D_2$; and $C P_1$, $C D_1$ are CONJUGATE SEMIDIAMETERS.

A straight line drawn parallel to $D_1 D_2$ (i. e. to the tangent at P_1) to meet $P_1 P_2$ is called an ORDINATE to the diameter $P_1 P_2$.

If this ordinate *pass through the focus* and is terminated both ways by the curve, it is called a PARAMETER to the diameter $P_1 P_2$.

The line drawn at right angles to the tangent at the point of contact is called the NORMAL.

If chords be drawn from any point in the circumference of the ellipse to the ends of a diameter, these two chords are called SUPPLEMENTAL CHORDS.

The AUXILIARY CIRCLE is the circle described on the axis major as diameter.

CONFOCAL ellipses are those which have the same foci.

SIMILAR ELLIPSES are those whose axes are proportionals.

Note.—Circles in the same plane are evidently projected onto any other plane as similar ellipses, and if the circles be concentric the ellipses will be concentric.

PROPOSITION VIII.

If the tangent at P meet the major axis produced at T, and P M be the ordinate; $C\,T\,.\,C\,M = C\,A^2$.

The tangent at P′ to the circle of projection cuts the diameter of which the major axis of the ellipse is the projection in T′.

Since C T T′ C′ is a parallelogram,

$$C'\,T' = C\,T;$$

and similarly

$$C'\,M' = C\,M, \quad C'\,A' = C\,A.$$

Since the angle at P′ is a right angle, from Eu. VI. 8, the triangles C′ P′ T′, C′ P′ M′ are similar;

$$\therefore\ C'\,T' : C'\,P' :: C'\,P' : C'\,M';$$

$$\therefore C'T' \cdot C'M' = C'P'^2 \text{ (Eu. vi. 16)}, = C'A'^2;$$

$$\therefore CT \cdot CM = CA^2.$$

PROPOSITION IX.

If the tangent at a point P meet the minor axis C B produced in t, and P n be perpendicular to that axis; $Ct \cdot Cn = CB^2$.

In the circle of projection, as in the last proposition,

$$C't' : C'P' :: C'P' : C'n';$$

$$\therefore C't' : C'B' :: C'B' : C'n';$$

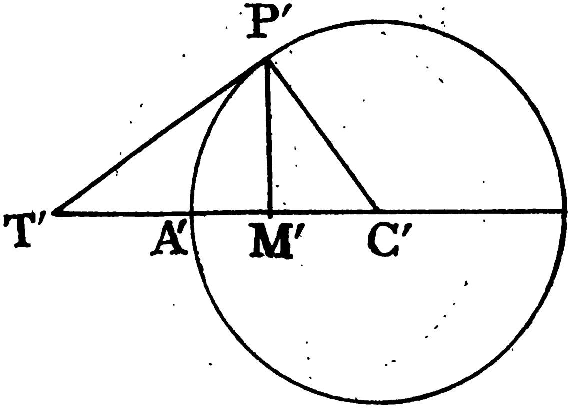
$$\therefore C't' \cdot C'n' = C'B'^2.$$

Also $\quad C't' : Ct :: C'B' : CB,$

$$C'n' : Cn :: C'B' : CB;$$

$$\therefore C't' \cdot C'n' : Ct \cdot Cn :: C'B'^2 : CB^2;$$

$$\therefore C't' \cdot C'n' : C'B'^2 :: Ct \cdot Cn : CB^2.$$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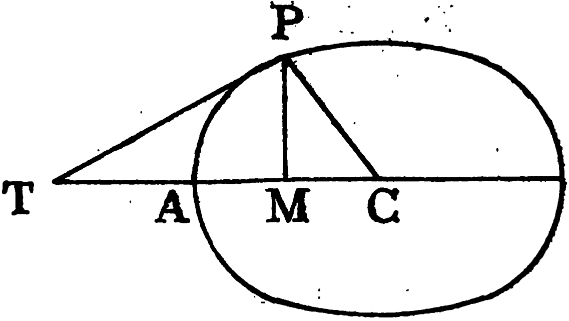
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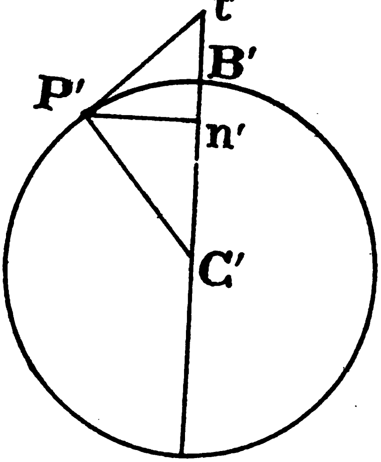
t′
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P′
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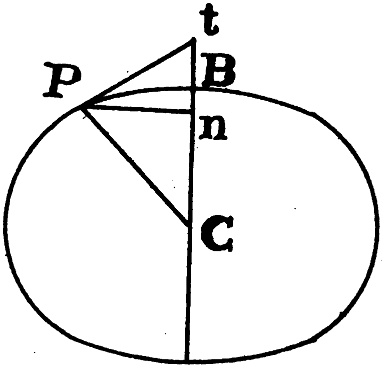
C′


t
B
P
n
C

But $$C' t' . C' n' = C' B'^2;$$

$$\therefore\ C t . C n = C B^2.$$

PROPOSITION X.

If from an external point O a pair of tangents, O Q, O R, be drawn to an ellipse, and C O be joined, meeting the chord Q R in V, and the ellipse in P; Q R will be bisected in V.

Let Q′ R′ be the circle of projection; O′ the point of which O is the projection.

∴ O′ R′, O′ Q′ are tangents to the circle.

In the circle, R′ Q′ is bisected in V′ (Appendix).

∴ the projection of R′ Q′, i. e.

R Q, is bisected in V

the projection of V′.

PROPOSITION XI.

Also the tangent at P will be parallel to Q R.

In the circle of projection,

$R'Q'$ is at right angles to $O'C'$;

so also is the tangent at P′ (Eu. III. 18).

$\therefore$ $R'Q'$ is parallel to the tangent at P′.

$\therefore$ R Q, the projection of $R'Q'$, is parallel to the projection of the tangent to the circle at P′ (Prop. II.), i. e. to the tangent to the ellipse at P.

PROPOSITION XII.

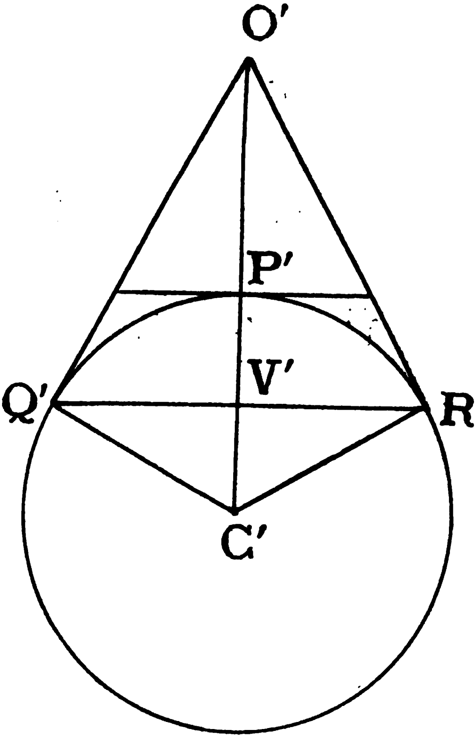
Also $CV : CP :: CP : CO$.

As in Propositions VIII. and IX.,

$$C'V' : C'Q' :: C'Q' : C'O';$$

$$\therefore C'V' : C'P' :: C'P' : C'O';$$

$$\therefore CV : CP :: CP : CO.$$


O′
P′
V′
Q′
R′
C′

PROPOSITION XIII.

If on the major axis of an ellipse as diameter, a circle be described, and the common ordinate M P Q be drawn meeting the ellipse in P, and the circle in Q, the tangent to the ellipse at the point P and the tangent to the circle at the point Q shall meet the major axis produced in the same point.

Let the tangent to the circle at Q meet the major axis in T, and the tangent to the ellipse at P meet it in U.

From Eu. VI. 8,

$$CM \cdot CT = CA^2.$$

From Proposition VIII.,

$$CM \cdot CU = CA^2.$$

$$\therefore CM \cdot CT = CM \cdot CU.$$

$$\therefore CT = CU;$$

i. e. T and U are coincident.

PROPOSITION XIV.

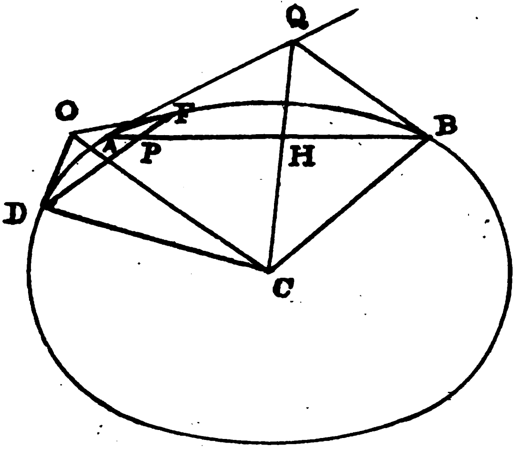
Chords to an ellipse are drawn through any fixed point within it; the points of intersection of the respective pairs of tangents drawn to the ellipse at the extremities of these chords lie on a straight line.

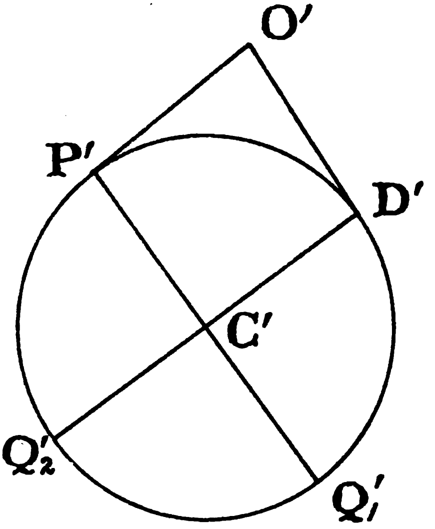
Let P be the point, P′ the corresponding point (i. e. the point of which P is the projection,) within the circle of projection.

It can be shown (Appendix) that the intersections of the respective pairs of tangents to the circle at the extremities of chords through P′ intersect on a straight line perpendicular to the straight line joining P′ to C′ the centre of the circle.

Each chord of the circle through P′ projects into a chord of the ellipse through P.

The tangents from any point Q′ to the ends of a chord of the circle project into tangents from a point Q, into which Q′ projects, to the ends of the chord of the ellipse into which that chord projects; and the straight line on which the points Q_1 Q_2 lie projects (Proposition I.) into a straight line.


Q
O
F
A
P
H
B
D
C


O′
P′
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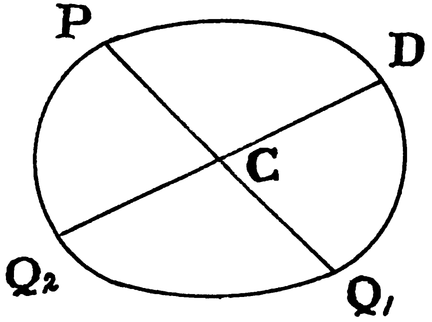
Q'_2
Q'_1


P
D
C
Q_2
Q_1

PROPOSITION XV.

Conjugate diameters of an ellipse are the projections of diameters of the circle of projection which are perpendicular to one another.

Let $P'Q_1'$, $D'Q_2'$ be the diameters of the circle of projection of which the conjugate diameters of the ellipse are projections.

By the supposition,

The tangents at $P'Q_1'$ are parallel to $D'Q_2'$;

but these tangents are perpendicular to $P'Q_1'$ (Eu. III. 18);

$\therefore$ $D'Q_2'$ is perpendicular to $P'Q_1'$.

PROPOSITION XVI.

In the ellipse, if C D be conjugate to C P, then will C P be conjugate to C D.

In the circle of projection; since

P′ O′, the tangent at P′, is perpendicular to C′ P′ (Eu. III. 18),

and by the supposition

is parallel to C′ D′ (note Prop. XV.);

∴ C′ D′ is perpendicular to C′ P′;

∴ (Eu. III. 18) C′ P′ is parallel to the tangent at D′;

∴ the projections of these are parallel—

the diameter C P to the tangent at D.

PROPOSITION XVII.

If CP, CD be conjugate semidiameters, PN, DR the ordinates of the points P, D; $CN^2 + CR^2 = CA^2$.

In the circle of projection,

$$\because P'C'O' + O'C'D' = \text{a right angle},$$

and $$P'C'N' + P'C'O' = \text{a right angle};$$

$$\therefore P'C'N' = O'C'D'.$$

Also $$P'C'N' + C'P'N' = \text{a right angle},$$

$$O'C'D' + D'C'R' = \text{a right angle};$$

$$\therefore C'P'N' = D'C'R',$$

and $$N'C'P' = C'D'R'.$$

$\therefore$ the triangles $N'C'P'$, $R'C'D'$ are similar;

$$\therefore N'C' : C'P' :: R'D' : C'D',$$

$$:: R'D' : C'P';$$

$$\therefore N'C' = R'D'.$$

$$\therefore\ C'D'^2 = C'N'^2 + C'R'^2;$$

$$\therefore\ C'A'^2 = C'N'^2 + C'R'^2.$$

$$\therefore\ CA^2 = CN^2 + CR^2.$$

PROPOSITION XVIII.

Also $NP^2 + DR^2 = BC^2$.

In the circle of projection,

$$C'N'^2 + N'P'^2 = C'P'^2 = C'A'^2;$$

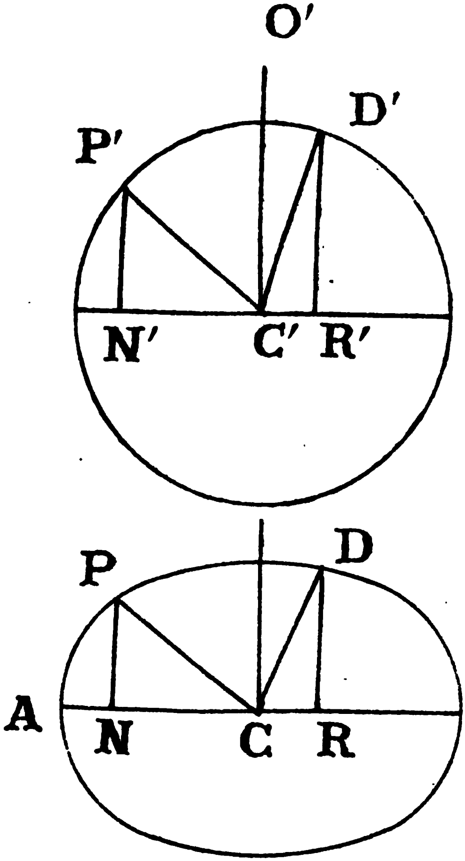
$$C'R'^2 + R'D'^2 = C'D'^2 = C'A'^2;$$

$$\therefore\ C'N'^2 + C'R'^2 + N'P'^2 + R'D'^2 = 2\,C'A'^2.$$

$$C'N'^2 + C'R'^2 = C'A'^2 \text{ (Prop. XVII.)};$$

$$\therefore\ N'P'^2 + R'D'^2 = C'A'^2 = C'B'^2.$$

$\therefore$ in the ellipse, $NP^2 + RD^2 = CB^2$ (Prop. IV.).


O′
D′
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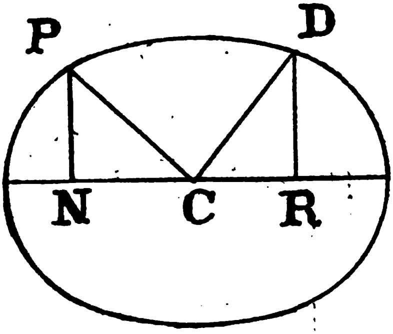
P
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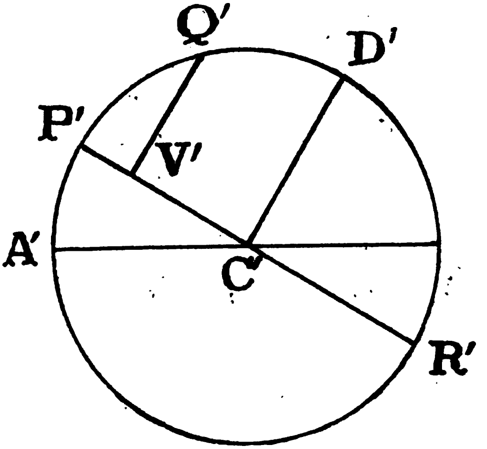
Q′
D′
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V′
A′
C′
R′

PROPOSITION XIX.

Also $CP^2 + CD^2 = AC^2 + BC^2$.

$$CP^2 + CD^2 = CN^2 + NP^2 + CR^2 + RD^2,$$
$$= CA^2 + CB^2 \text{ (Props. XVII., XVIII.)}.$$

PROPOSITION XX.

If CD *be a diameter conjugate to* CR, QV *an ordinate to* PCR;

$$QV^2 : PV.RV :: CD^2 : CP^2.$$

In the circle of projection,

$$Q'V'^2 = P'V'.R'V' \text{ (Eu. vi. 13)},$$
$$C'D' = C'P';$$
$$\therefore Q'V'^2 : C'D'^2 :: P'V'.R'V' : C'P'^2.$$

In projection, $Q'V'$, $C'D'$ are reduced in the same proportion (Prop. IV.);

$$\therefore QV^2 : CD^2 :: P'V'.R'V' : C'P'^2.$$

Similarly $P'V'$, $R'V'$, $C'P'$ are reduced in the same proportion;

$$\therefore QV^2 : CD^2 :: PV.RV : CP^2.$$

PROPOSITION XXI.

If CP, CD be conjugate diameters of an ellipse, and PF be drawn at right angles to CD, meeting CA in G,

$$PF.PG = BC^2.$$

From P draw PMW perpendicular to the major axis, meeting it in M, and meeting DC in W.

Since the angles at M and G are right angles, a circle can be described about MGFW (Eu. III. 22);

$$\therefore PM.PW = PG.PF \text{ (Eu. III. 36)}.$$

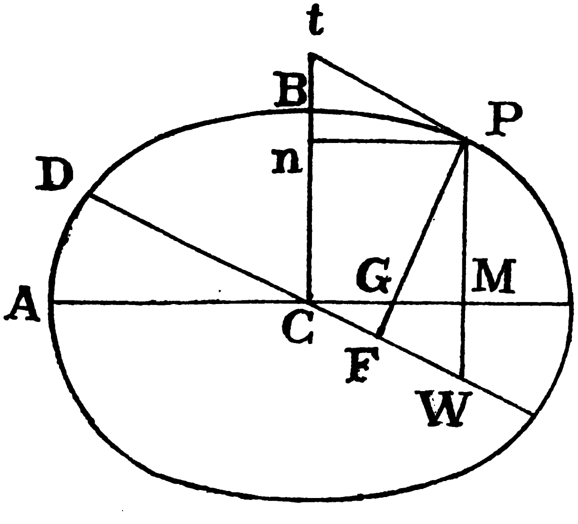
Draw P*n* perpendicular to BC, and let the tangent at P meet CB, produced, in *t*.

$$PM = Cn;$$

and, since PWC*t* is a parallelogram,

$$PW = Ct.$$

$$\begin{aligned} \therefore PF.PG &= PM.PW, \\ &= Cn.Ct, \\ &= BC^2 \text{ (Prop. IX.)}. \end{aligned}$$


t
B
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M
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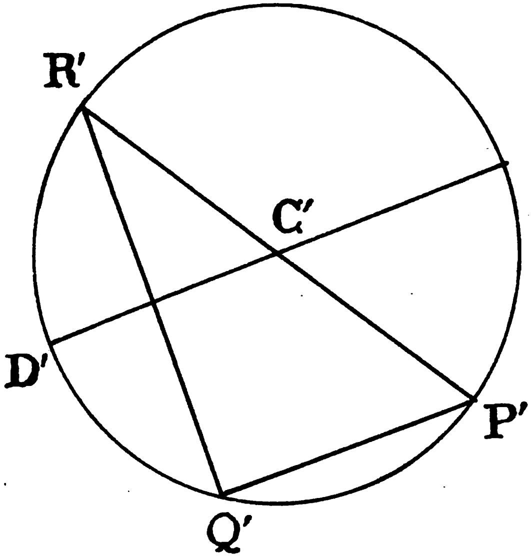

R′
C′
D′
P′
Q′

PROPOSITION XXII.

If a chord and a diameter of an ellipse are parallel, the supplemental chord is parallel to the conjugate diameter.

Let P′ Q′ be the corresponding chord in the circle of projection, C′ D′ the diameter parallel to it; P′ C′ R′ will be the chord to the other end of which the supplemental chord Q′ R′ is drawn.

P′ Q′ R′ is a semicircle;

∴ Q′ R′ is perpendicular to P′ Q′;

∴ Q′ R′ is perpendicular to C′ D′;

∴ Q′ R′ is parallel to the tangent at D′;

∴ Q R is parallel in the ellipse to the diameter conjugate to C D.

PROPOSITION XXIII.

If P be any point on the circumference of an ellipse, the sum of the focal distances SP, SP = *the major axis.*

Draw the ordinate PN meeting the major axis in N; also PM, $\mathrm{P}M$ parallel to the major axis and ∴ perpendicular to the directrices, meeting them in M, M.

$$\mathrm{SP} : \mathrm{PM} :: S\mathrm{P} : \mathrm{P}M;$$

$$\therefore\ \mathrm{SP} : \mathrm{NX} :: S\mathrm{P} : \mathrm{N}X;$$

alternando, $\mathrm{SP} + S\mathrm{P} :: \mathrm{NX} : \mathrm{N}X$;

componendo, $\mathrm{SP} + S\mathrm{P} : \mathrm{SP} :: \mathrm{NX} + \mathrm{N}X : \mathrm{NX}$;

alternando, $\mathrm{SP} + S\mathrm{P} : \mathrm{X}X :: \mathrm{SP} : \mathrm{NX}$;

$$:: \mathrm{CS} : \mathrm{CA};$$

$$:: \mathrm{CA} : \mathrm{CX};$$

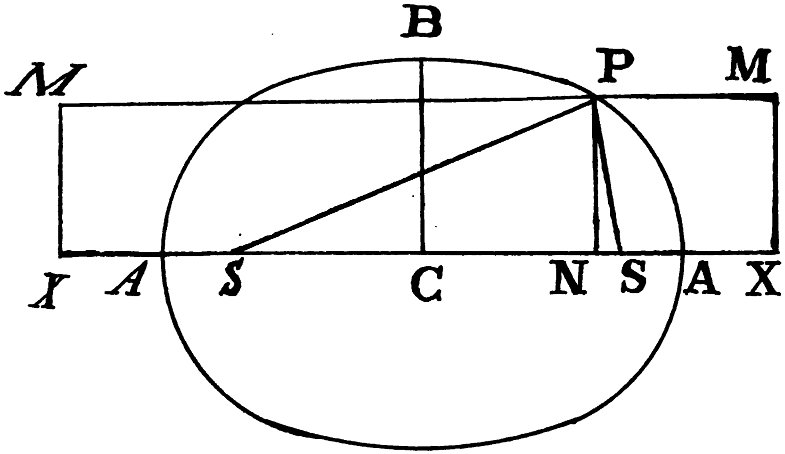
$$:: 2\mathrm{CA} : 2\mathrm{CX};$$

$$:: \mathrm{A}A : \mathrm{X}X;$$

$\therefore\ \mathrm{SP} + S\mathrm{P} = \mathrm{A}A$, which is the major axis.

Corollary. $\mathrm{SB} + \mathrm{S'B} = \mathrm{A}A$;

$\therefore\ \mathrm{SB} = \mathrm{CA}$, or a, i. e. half the major axis.


B
M
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I
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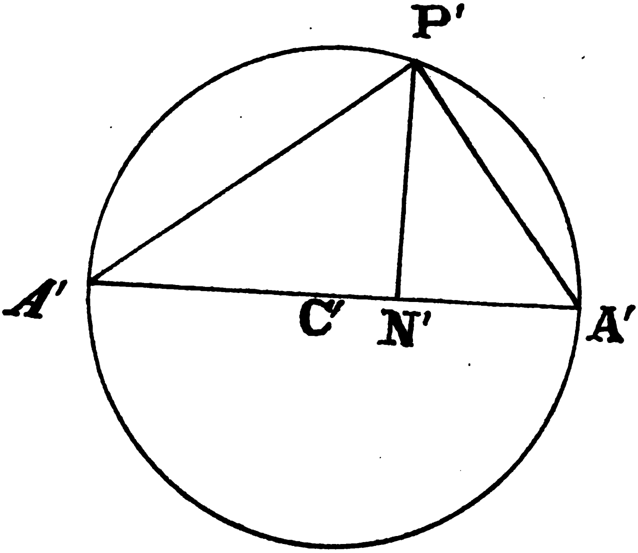
S
C
N
S
A
X


P′
A′
C′
N′
A′

PROPOSITION XXIV.

If P N be the ordinate of an ellipse, meeting the major axis in N,

$$PN^2 : AN \,.\, AN :: b^2 : a^2;$$

where b denotes half the axis minor, a half the axis major.

In the circle of projection,

$$A'N' \,.\, A'N' = P'N'^2 \text{ (Eu. vi. 13)};$$

$$PN^2 : P'N'^2 :: b^2 : a^2 \text{ (from Prop. IV.)};$$

$$\therefore PN^2 : AN \,.\, AN :: b^2 : a^2.$$

PROPOSITION XXV.

If $S\,L$ be the semi-latus-rectum,

$$a \,.\, S\,L = b^2.$$

From Proposition XXIV.,

$$\mathrm{S\,L}^2 : \mathrm{A\,S} \,.\, A\,\mathrm{S} :: b^2 : a^2.$$

From Eu. II. 5,

$$\mathrm{A\,S} \,.\, A\,\mathrm{S} = a^2 - \mathrm{C\,S}^2.$$

From corollary of Proposition XXIII.,

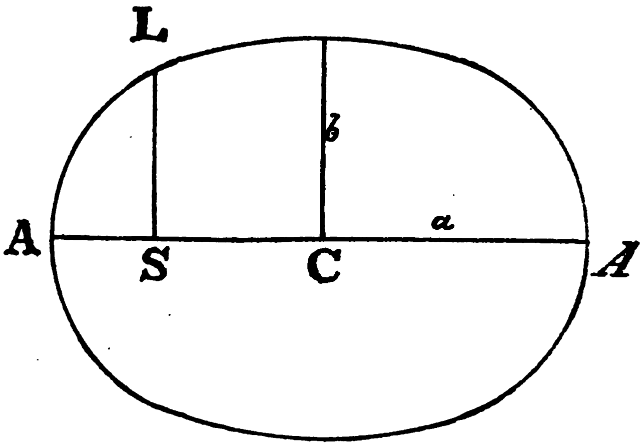
$$\mathrm{C\,S}^2 = a^2 - b^2.$$

$$\therefore\ \mathrm{S\,L}^2 : a^2 - (a^2 - b^2) :: b^2 : a^2;$$

$$\therefore\ \mathrm{S\,L}^2 : b^2 :: b^2 : a^2;$$

$$\therefore\ \mathrm{S\,L} : b :: b : a;$$

$$\therefore\ a \,.\, \mathrm{S\,L} = b^2 \text{ (Eu. VI. 17).}$$


L
b
a
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S
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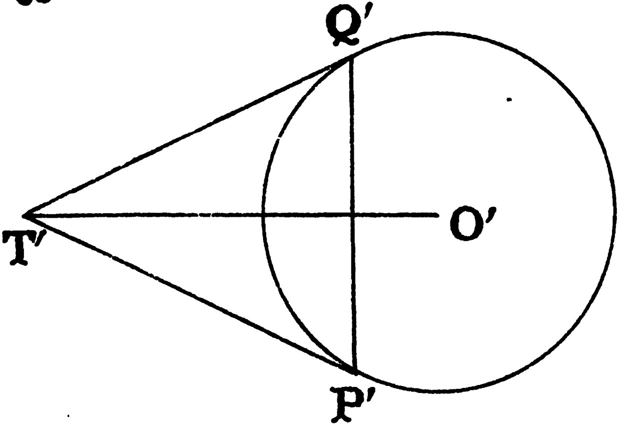
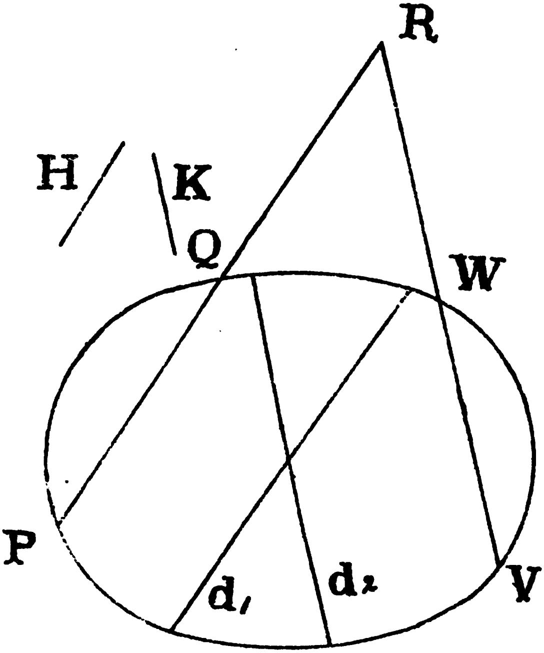
A


Q′
T′
O′
P′


R
H
K
Q
W
P
d₁
d₂
V

PROPOSITION XXVI.

Tangents at the extremities of any chord of an ellipse meet on the diameter which bisects that chord.

Let $P'Q'$ be the circle of projection; $P'Q'$ the chord of the circle of projection of which the chord PQ is the projection; and $P'T'$, $Q'T'$ the tangents to the circle at the ends of this chord.

$O'T'$ bisects this chord (Appendix);

∴ OT bisects the chord PQ (Prop. III.).

PROPOSITION XXVII.

If through any point without an ellipse two straight lines be drawn parallel to two given straight lines so as to meet the curve, the rectangles of the segments shall be to one another in the constant ratio of the squares of the diameters parallel to them.

Let H, K be the two given straight lines; let PR be parallel to H, meeting the ellipse in P, Q; VW parallel to K, meeting the ellipse in V, W, and intersecting PQR in R.

Let $P'Q'W'V'$ be the circle of projection; H', K', $P'Q'R'$, $V'W'R'$, the straight lines in its plane of which H, K, PQR, VWR, are the projections; and let $D_1'D_2'$ be diameters of the circle parallel to H', K' (∴ to $P'R_1'$, $V'R'$,) respectively.

Since $D_1' = D_2'$,

and $R'Q' . R'P' = R'W . R'V'$ (Eu. III. 36);

$R'Q' . R'P' : D_1'^2 :: R'W' . R'V' : D_2'^2$;

i. e. $R'Q' . R'P' : R'W' . R'V' :: D_1'^2 : D_2'^2$.

$\therefore$ in the projection (note Proposition IV. and Eu. VI. 23),

$$RQ . RP : RW . RV :: D_1^2 : D_2^2;$$

and the ratio of D_1 to D_2, and therefore of D_1^2 to D_2^2, is constant, for they are given fixed diameters, since they are respectively parallel, by hypothesis, to the given fixed straight lines H, K.

PROPOSITION XXVIII.

If a diameter bisects all chords parallel to a second diameter, the second diameter bisects all chords parallel to the first.

Let P Q be a diameter bisecting all chords parallel to the diameter W V.

Let P′ Q′, W′ V′ be the corresponding chords in the circle of projection.

If P′ Q′ bisects a chord, it is perpendicular to it (Eu. III. 3);

$\therefore$ P′ Q′ is perpendicular to W′ V′;

since this is, by hypothesis, parallel to the bisected chords.

$\therefore$ W′ V′ is perpendicular to all chords parallel to P′ Q′;

$\therefore$ it bisects them;

$\therefore$ the projection of W′ V′ bisects the projections of these chords.

PROPOSITION XXIX.

The area of the parallelogram described about a given ellipse, at the extremities of any pair of conjugate diameters, is constant.

Let the parallelogram about the ellipse be divided by ordinates perpendicular to the major axis into a number of narrow strips.

Let $P_1 Q_1 Q_2 P_2$ be one of these strips.

Through $Q_1 Q_2$ respectively draw $Q_1 E_1$, $Q_2 E_2$ parallel to N R the major axis.

The rectangle $E_1 Q_2$ differs from the figure $P_1 Q_2$ by the sum of the triangles $P_1 E_1 Q_1$, $P_2 E_2 Q_2$; i.e. it is too small by a proportion of itself represented by

$$\frac{P_1 E_1 Q_1 + P_2 E_2 Q_2}{E_1 Q_2}.$$

Divide N R into two parts in I.

Through I draw $J_1 K_1 I K_2 J_2$, another ordinate parallel to $P_1 P_2$ or $Q_1 Q_2$; and draw $J_1 H_1$, $J_2 H_2$ parallel to N R.

Since $H_1 J_2 + K_1 Q_2$ is greater than $E_1 Q_2$, and $PHJ + JKQ$ is less than PEQ, the sum of the rectangles $H_1 J_2$, $K_1 Q_2$, differs from the figure $P_1 Q_2$ by a less proportion than in the previous case.

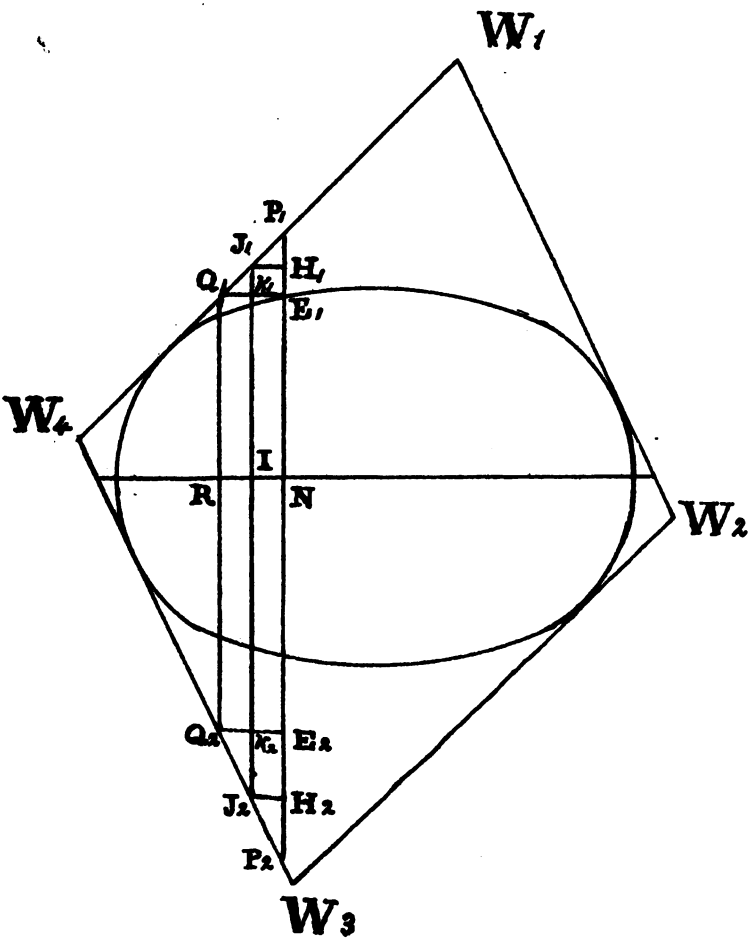
It thus appears that if we assign any proportion of error, we can, by increasing the number of rectangles, make their sum differ from the figure $P_1 Q_2$ by less than this assigned amount; or, we may take the sum of the rectangles to be the area of the figure $P_1 Q_2$.

If now we take the sum of all such rectangles, from W_4 to W_2, we shall have the area of the parallelogram $W_1 W_3$ about the ellipse.

Similarly, if we take the sum of the strips of which these are the projections, we shall have the area of the figure of which $W_1 W_3$ is the projection; this will, evidently, be the parallelogram about the circle of projection (Propositions II., VI.,) at the ends of diameters at right angles to one another (Proposition XV.); i. e. (note Eu. IV. 7) the square which circumscribes the circle—the square on the diameter.

From Proposition IV., the area of any one of the rectangular elements of the figure $W_1 W_3$ bears to the corresponding element of which it is the projection a certain ratio (namely, that of CB to C′B′, or of a to b,) independent of the position of the conjugate diameters $W_1 W_3$, $W_2 W_4$:

∴ the figure $W_1 W_3$: square on major axis, in this ratio; that is to say, since this square is constant, the area of $W_1 W_3$ is constant.


W1
P1
J1
H1
Q1
K1
E1
W4
I
R
N
W2
Q2
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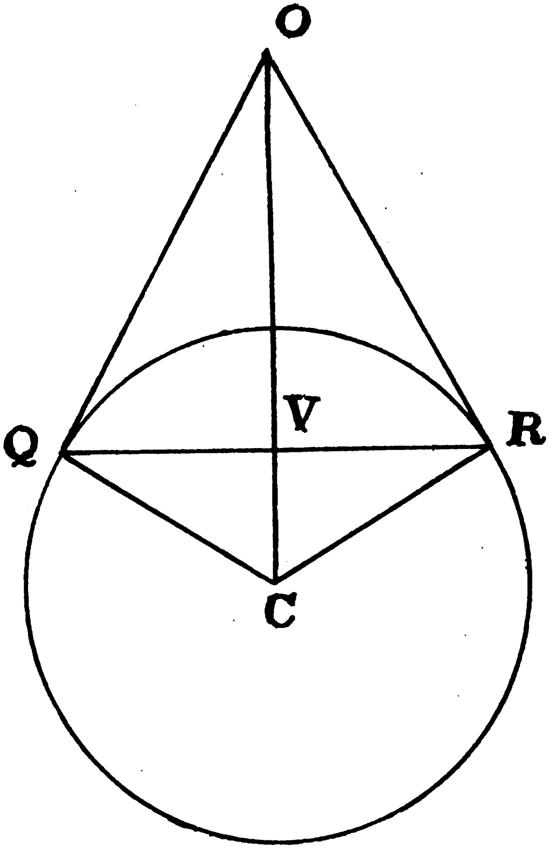
E2
J2
H2
P2
W3

56



APPENDIX.

PROPOSITION V.

The ratio SP : NP is called the excentricity of the ellipse, and is denoted in algebraical or analytical geometry by the symbol e: thus, a denoting the semi-axis-major, b the semi-axis-minor,

$$\mathrm{CS} = ea, \qquad a = e\,.\,\mathrm{CO};$$

$$\therefore\ b^2 = a^2 - \mathrm{CS}^2 = a^2(1 - e^2).$$

The student of trigonometry will note that if from any point P′ on the circumference of the circle of projection a perpendicular P′M′ be let fall onto that diameter of which the major axis is the projection, and PM be the corresponding ordinate of the ellipse,

$$\frac{\mathrm{PM}}{\mathrm{P'M'}} = \cos.\,\theta, \text{ or } \mathrm{PM} = \mathrm{P'M'}\cos.\,\theta,$$

θ being the angle between the two planes.

Also, $\frac{b}{a} = \cos.\ \theta$, $\therefore$ if $\mathrm{P'M'} = y'$, $\mathrm{CM} = x$, $\mathrm{PM} = y = y' \cos.\ \theta$,

$$\mathrm{P'M'^2} + \mathrm{C'M'^2} = \mathrm{P'M'^2} + \mathrm{CM^2} = \mathrm{CP^2} = a^2, \text{ or } x^2 + \frac{y^2}{\cos.^2\theta} = a^2;$$

$$\therefore\ y^2 = \frac{b^2}{a^2}(a^2 - x^2)$$

the equation to the ellipse, referred to the centre as origin, of analytical conics.

The projection of a figure may be practically depicted by drawing the figure on a piece of glass and then exposing it to sunlight so that the shadow may fall onto the plane required; the sun's rays, on account of his great distance, being in effect parallel, this shadow will be the orthogonal projection. If a candle flame be used it will be the conical projection—assuming the flame to be a point.

PROPOSITIONS X., XXVI.

In the triangles CQO, CRO,

$$\mathrm{CQ} = \mathrm{CR}, \qquad \mathrm{CO} \text{ is common};$$

the angles at Q, R are right angles,

$$\therefore\ \mathrm{OQ^2} + \mathrm{CQ^2} = \mathrm{OR^2} + \mathrm{CR^2},$$

$$\therefore\ \mathrm{OQ} = \mathrm{OR};$$

∴ the triangles are equal in all respects (Eu. I. 8);

∴ the angle QCV = RCV.

In the triangles CQV, CRV,

$$\mathrm{CQ} = \mathrm{CR}, \qquad \mathrm{CV} \text{ is common},$$

the angle QCV = RCV;

∴ (Eu. I. 4) QV = VR,

i.e. QR is bisected in V.

PROPOSITION XIV.

Let P be the fixed point through which the chords are drawn.

Join C P, and produce.

Through P draw D P F perpendicular to C P.

Let the tangents at D, F meet C P produced in O; through O draw O Q perpendicular to C O.

Let A B be any other chord through P; bisect A B in H; join C H, and produce it to meet O Q in Q.

Join Q A, Q B.

$$\begin{aligned} AP \cdot PB &= DP \cdot PF \text{ (Eu. III. 35)}, \\ &= DP^2 \text{ (since } DP = PF\text{)}, \\ &= CP \cdot PO \text{ (Eu. VI. 8)}; \end{aligned}$$

∴ a circle can be described about C A O B;

$$\begin{aligned} \therefore\ \angle AOC &= \angle ABC, \\ &= BAC \text{ (since } CA = CB\text{)}. \end{aligned}$$

P O Q is a right angle, and C A B + A C Q = a right angle (since A O C = C A B);

$$\therefore\ \{AOC + QOC\} + ACQ = 2 \text{ right angles};$$

∴ a circle can be described about A O Q C (Eu. III. 22).

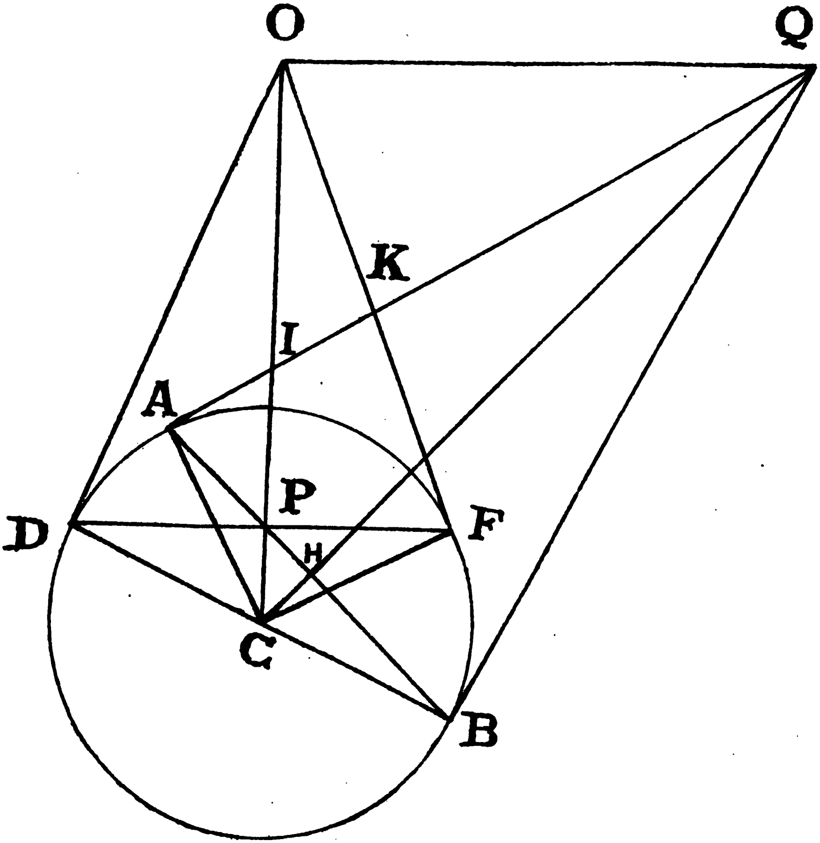
$$\therefore\ OQA = ACO;$$

but $OIQ + OQA$ = a right angle,

∴ A I C (= O I Q from Eu. I. 15) + A C O = a right angle;

∴ I A C is a right angle;

∴ Q A is the tangent at A.


O
Q
K
I
A
P
D
F
H
C
B

EXAMPLES.

1. Every diameter is bisected in C the centre of the ellipse.

2. The tangent to an ellipse at P meets the axis major produced in T; at T a perpendicular to the axis major is erected, and A P, *A* P intersect this perpendicular in K, *K*: show that T K = T *K*.

3. Show that an ellipse can be projected into a circle.

4. If a diameter bisects all chords parallel to a second diameter, this second diameter bisects all chords parallel to the first.

5. C P is a semidiameter; A Q K, drawn parallel to C P, meets the curve in Q, and the minor axis produced in K: show that A K . A Q $= \mathrm{CP}^2$.

6. Show that the parallelogram formed by drawing tangents at the extremities of conjugate diameters is the least which can be described about the ellipse.

7. Show that of all pairs of conjugate diameters, that of whose squares the sum is least consists of the axis major and axis minor of the ellipse.

8. P is a given point on the circumference of an ellipse: find the least triangle which can be inscribed with P for a vertex.

9. WP_1, WP_2 are tangents to an ellipse; CQ_1, CQ_2 are radii (i.e. semidiameters) respectively parallel to them: prove that $\mathrm{P}_1\mathrm{P}_2$, $\mathrm{Q}_1\mathrm{Q}_2$ are parallel.

10. The tangent at P meets the axis major produced in T; P N is the perpendicular from P upon the axis major: prove that

$$\mathrm{AN}\,.\,A\mathrm{N} : \mathrm{AT}\,.\,A\mathrm{T} :: \mathrm{CN} : \mathrm{CT}.$$

11. C P, C D are conjugate semidiameters; PG_1, DG_2 are drawn perpendicular to C D, C P, respectively, meeting the axis major in G_1, G: prove that

$$\mathrm{PG}_1^2 + \mathrm{PG}_2^2 : \mathrm{AB}^2 :: b^2 : a^2.$$

12. The tangent at any point P meets a pair of conjugate diameters produced in $\mathrm{K}_1\,\mathrm{K}_2$: prove that

$$\mathrm{PK}_1\,.\,\mathrm{PK}_2 = \mathrm{CD}^2,$$

where C D is the semidiameter parallel to the tangent—i.e. conjugate to C P.

13. C P, C Q are two conjugate semidiameters; the tangents at P Q meet one another at K: prove that the

Areas of the triangles C P K, C Q K are equal.

14. Prove that if C P, C Q be any two semidiameters; and the tangents at P, Q meet C Q and C P respectively produced in $\mathrm{H}_1\,\mathrm{H}_2$; the areas of the triangles CPH_1, CQH_2 are equal.

15. C P, C D are two equal and conjugate semidiameters; the tangent at the vertex A meets them produced in H_1, H_2 respectively: show that

$$\mathrm{PD}^2 = 2\,\mathrm{AH}_1^2 = 2\,\mathrm{AH}_2^2.$$

16. If $2a$, $2b$ denote the lengths of the axes of an ellipse; and the area of a circle of radius a is πa^2: the area of the ellipse $= \pi a b$.

17. Chords of an ellipse are drawn through any fixed point without it: show that the points of intersection of the respective pairs of tangents, drawn to the ellipse at the points where these chords meet it, lie on a straight line.

18. If from any points in a straight line a pair of tangents be drawn to an ellipse, the chords of contact will all pass through a fixed point.

19. $P_1 P_2$ is a chord parallel to the major axis; $P_1 Q_1$, $P_1 Q_2$ are chords making equal angles with the major axis: prove that $Q_1 Q_2$ is parallel to the tangent at P_1.

20. $P_1 P_2$ at right angles to the axis major cuts four concentric co-axial ellipses in $P_1 P_2 P_3 P_4$; CD_1, CD_2, CD_3, CD_4 are the semidiameters conjugate to CP_1, CP_2, CP_3, CP_4: show that

$D_1 D_4$ is parallel to the axis major and passes through $D_2 D_3$.

21. CP_1, CP_2 are conjugate semidiameters; CQ_1, CQ_2 are another pair of conjugate semidiameters: show that the areas of the rectilineal triangles $CP_1 Q_1$, $CP_2 Q_2$ are equal.

22. The greatest ellipse which can be inscribed in any parallelogram touches its sides at their middle points.

23. $P_1 C P_2$, $Q_1 C Q_2$ are conjugate diameters; a triangle is inscribed in the ellipse with its vertex at P_1, and its sides bisected by $P_1 C P_2$, $Q_1 C Q_2$: prove that this is the greatest triangle that can be inscribed in the ellipse.

24. CP, CD are conjugate semidiameters; PG is the normal at P, meeting the axis major in G; PN_1, DN_2 are perpendiculars from P, D upon the axis major: show that the triangles

PGN_1, DGN_2 are similar.

25. Tangents are drawn at the ends of the axis major and axis minor, forming a rectangle about the ellipse: show that its diagonals coincide with equal conjugate diameters.

26. Q is any point on the circumference of an ellipse; BQ are joined; and AQ_2 is drawn parallel to BQ meeting the curve in Q_2: show that

CQ, CQ_2 are conjugate.

27. Q is any point on the circumference of an ellipse; $P_1 P_2$, $D_1 D_2$ are any pair of conjugate diameters; QP_1, QP_2 meet CD_1 produced in H, K: show that

$$CH \cdot CK = CD_1^2.$$

28. The tangent at the vertex cuts any two conjugate diameters in H, K: show that

$$AH \cdot AK = b^2.$$

29. $P_1 C P_2$ is any diameter, CD its conjugate; $Q_1 Q_2$ are any two points on the ellipse; the tangents at Q_1, Q_2 intersect at H; $P_1 Q_1$, $P_2 Q_2$ intersect in K: prove that HK is parallel to CD.

30. $P_1 P_2$, $Q_1 Q_2$ make equal angles with the axis major: show that $P_1 Q_1$, $P_2 Q_2$ are equally inclined to that axis.

31. P is any point within an ellipse: show that the middle points of all chords through P lie on a similar ellipse.

32. A chord $P_1 Q_1 Q_2 P_2$ is drawn cutting one of two concentric similar ellipses in $P_1 P_2$, the other in $Q_1 Q_2$: show that

$$P_1 Q_1 = Q_2 P_2.$$

33. The tangent at P meets the axis major produced in T, CP produced meets the tangent at the adjacent vertex in H: show that

HT is parallel to AP.

34. P is any point on the ellipse; AP, AP cut the minor axis produced in HK: show that

$$AH \cdot AP + \mathit{A}K \cdot AP = 2AB^2.$$